§3.2 Properties of Determinants

Question: How do row operations affect determinants?

Theorem

Let A be a square matrix and B a matrix ~~exept~~ obtained after applying an elementary row operation on A.

a) If a multiple of one row is added to another to produce B, then $\det A = \det B$

b) If two rows are interchanged to produce B, then $-\det A = \det B$

c) If one row of A is multiplied by constant k to produce B, then $\det B = k \cdot (\det A)$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det A = 4 - 6 = -2$

a) $A \xrightarrow{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$ $\det \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix} = 8 - 10$

$= -2$

$= \det A$

b) $A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\quad \det \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = 6 - 4 = 2$

$$= -\det A$$

c) $A \xrightarrow{-3R_2} \begin{bmatrix} 1 & 2 \\ -9 & -12 \end{bmatrix}$ $\quad \det \begin{bmatrix} 1 & 2 \\ -9 & -12 \end{bmatrix} = -12 + 18$

$$= 6$$

$$= -3 \cdot \det A$$

This is useful because we can row reduce A to a simpler echelon form and keep track how the determinant changes.

Defn: A square matrix is upper (lower) triangular if the entries below (above) the main diagonal are zero

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix} \qquad \left(\begin{bmatrix} a_{11} & & 0 \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \right)$$

Theorem

The determinant of an upper or lower triangular matrix is the product of the entries on the main diagonal:

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

Notice the echelon form of a square matrix is upper triangular!

<u>Theorem</u>
Suppose A is a square matrix. A is invertible if and only if $\det A \neq 0$.

<u>Proof</u>
Suppose E is an echelon form of A. Then $A \sim E$ and A is invertible if and only if $E \sim I_n$. By the invertible matrix theorem this is equivalent to saying E has n pivots, i.e. the diagonal entries are nonzero. Thus $\det E = e_{11} e_{22} \cdots e_{nn} \neq 0$

since $A \sim E$ $\quad \det A = \underset{\neq 0}{(\text{some number})} \; \underset{\neq 0}{\det E}$

so $\det A \neq 0$

<u>Corollary</u>
If A is a square matrix, the columns of A are linearly independent if and only if $\det A \neq 0$.

Example

Compute $\det A$ where $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ by row reduction.

$$\overset{\det A}{A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & -1 \\ -2 & 1 & 2 \end{bmatrix}} \xrightarrow[2R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \overset{\det A}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -7 \\ 0 & 1 & 8 \end{bmatrix}} \xrightarrow{R_2 \leftrightarrow R_3} \overset{-\det A}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 4 & -7 \end{bmatrix}}$$

$$\xrightarrow[\to R_3]{-4R_2 + R_3} \overset{-\det A}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & -39 \end{bmatrix}} \xrightarrow{-\frac{1}{39} R_3} \overset{-\frac{1}{39}(-\det A)}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}} = E$$

$$-\frac{1}{39}\left(-\det A\right) = 1 \cdot 1 \cdot 1$$

$$\frac{1}{39} \det A = 1$$

$$\boxed{\det A = 39}$$

In particular $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ is invertible

Theorem

If A is an $n \times n$ square matrix, then $\det A = \det A^T$.

Proof

A cofactor expansion along row k (of A) is the same as the cofactor expansion along column k of A^T. (Uses what's called mathematical induction).

corollary

If A is a square matrix, the rows of A are linearly independent if and only if $\det A \neq 0$.

Theorem

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B)$$

In particular, AB is invertible if and only if both A and B are invertible.

Warning!

In general, $\det(A+B) \neq \det(A) + \det(B)$

Example

compute $\det(A^6)$ Where $A = \begin{bmatrix} 2 & -4 & 0 \\ -6 & 1 & -1 \\ 3 & 4 & 1 \end{bmatrix}$

$$\det(A^6) = \det(\underbrace{A \cdots A}_{6 \text{ times}}) = (\det(A))^6$$

$$\det A = 2 \det\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} - (-4) \det\begin{bmatrix} -6 & -1 \\ 3 & 1 \end{bmatrix} + 0$$

$$= 2(1+4) + 4(-6+3)$$

$$= 10 - 12$$

$$= -2$$

$$\det(A^6) = (-2)^6 = \boxed{64}$$